

$$1) a) z = f(x, y) = xy \text{ w/ } x^2 + y^2 \leq 1$$

$$f_x(x, y) = y, f_y(x, y) = x$$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1+y^2+x^2} dA = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1+r^2} dr = 2\pi \left[\frac{(1+r^2)^{3/2}}{3} \right]_0^1 = \frac{2\pi}{3} (2\sqrt{2} - 1) \end{aligned}$$

$$b) \vec{r}_u = \langle 2u, v, 0 \rangle, \vec{r}_v = \langle 0, u, v \rangle$$

$$\text{Then } \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2 \hat{i} - 2uv \hat{j} + 2u^2 \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2 + 2u^2)^2} = v^2 + 2u^2$$

$$\begin{aligned} \text{Then, } A(S) &= \iint_D |\vec{r}_u \times \vec{r}_v| dA = \int_0^1 \int_0^2 v^2 + 2u^2 dv du = \int_0^1 \left[\frac{1}{3} v^3 + 2u^2 v \right]_{v=0}^2 du = \int_0^1 \left(\frac{8}{3} + 4u^2 \right) du = \\ &= \left[\frac{8}{3} u + \frac{4}{3} u^3 \right]_0^1 = \frac{8}{3} + \frac{4}{3} = 4 \end{aligned}$$

2) $S : x^2 + y^2 + z^2 = 4, z \geq 0$, so parametrization is given by

$$\vec{r}(\phi, \theta) = \langle 2\sin\phi \cos\theta, 2\sin\phi \sin\theta, 2\cos\phi \rangle, \quad 0 \leq \theta \leq 2\pi, \quad \underbrace{0 \leq \phi \leq \pi/2}_{\text{Since } z \geq 0}$$

• From lecture notes we know $|\vec{r}_\phi \times \vec{r}_\theta| = 4\sin\phi$

Then,

$$\begin{aligned} \iint_S x^2 z + y^2 z ds &= \iint_S z(x^2 + y^2) ds = \int_0^{2\pi} \int_0^{\pi/2} \underbrace{2\cos\phi}_z \underbrace{(4\sin^2\phi)}_{x^2+y^2} \cdot \underbrace{4\sin\phi}_{ds} d\phi d\theta \\ &= 32 \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3\phi \cos\phi d\phi = 64\pi \left[\frac{\sin^4\phi}{4} \right]_0^{\pi/2} = 16\pi \end{aligned}$$

3) Here S consists of three pieces

S_1 : the surface of the cylinder

S_2 : plane $x+y=2$ (the front)

S_3 : plane $y=0$ (the back)

S_1 : parametric equations given by $\vec{r}(\theta, y) = \langle \sin\theta, y, \cos\theta \rangle$

Then,

$$\vec{F}(\vec{r}(\theta, y)) = \langle \sin\theta, y, 5 \rangle$$

$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{vmatrix} = \sin\theta \hat{i} + \cos\theta \hat{k}$$

$$\begin{aligned} \text{Then } \iint_S \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^{2-\sin\theta} (2\sin^2\theta + 5\cos\theta) dy d\theta = \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta \cos\theta) d\theta \\ &= \int_0^{2\pi} (1 + \cos 2\theta + 10\cos\theta - \sin\theta(1 - \cos^2\theta) - 5\sin\theta \cos\theta) d\theta = \\ &= \left[\theta + \frac{\sin 2\theta}{2} + 10\sin\theta - \cos\theta - \frac{\cos^3\theta}{2} - \frac{5\sin^2\theta}{2} \right]_0^{2\pi} = 2\pi \end{aligned}$$

On S_2 $\vec{F}(\vec{r}(x, z)) = x\hat{i} + (2-x)\hat{j} + 5\hat{k}$ and $\vec{r}_x \times \vec{r}_z = \hat{i} + \hat{j}$

$$\text{Then, } \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \leq 1} [x + (2-x)] dA = 2\pi$$

On S_3 : $\vec{F}(\vec{r}(x, z)) = x\hat{i} + 5\hat{k}$, $\vec{r}_x \times \vec{r}_z = -\hat{j}$

$$\text{Then } \iint_S \vec{F} \cdot d\vec{S} = \iint_S 0 dS = 0$$

$$\text{Finally, } \iint_S \vec{F} \cdot d\vec{S} = 2\pi + 2\pi + 0 = 4\pi$$

4) If S is given by $y = h(x, z)$, then parametric equations are given by
 $x = x$, $y = h(x, z)$, $z = z$.

$$\text{Then } \vec{r}_x(x, z) = \hat{i} + \frac{\partial h}{\partial x} \hat{j}, \quad \vec{r}_z = \frac{\partial h}{\partial z} \hat{j} + \hat{k}$$

$$\text{Thus, } \vec{r}_x \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{\partial h}{\partial x} & 0 \\ 0 & \frac{\partial h}{\partial z} & 1 \end{vmatrix} = \frac{\partial h}{\partial x} \hat{i} - \hat{j} + \frac{\partial h}{\partial z} \hat{k}$$

$$\text{Then } \hat{n} = \frac{\vec{r}_x \times \vec{r}_z}{|\vec{r}_x \times \vec{r}_z|} = \frac{\frac{\partial h}{\partial x} \hat{i} - \hat{j} + \frac{\partial h}{\partial z} \hat{k}}{\sqrt{h_x^2 + 1 + h_z^2}} \quad \left(\text{Note that } \hat{n} \text{ is pointing to the left} \right)$$

Let $\vec{F} = \langle P, Q, R \rangle$, then

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \hat{n} \, dS = \iint_D \langle P, Q, R \rangle \cdot \frac{1}{\sqrt{h_x^2 + 1 + h_z^2}} \langle \frac{\partial h}{\partial x}, -1, \frac{\partial h}{\partial z} \rangle \underbrace{\sqrt{h_x^2 + 1 + h_z^2} \, dA}_{dS} \\ &= \iint_D \left(P \frac{\partial h}{\partial x} - Q + R \frac{\partial h}{\partial z} \right) dA \end{aligned}$$

$$5) u(x, y, z) \propto \frac{1}{\sqrt{x^2 + y^2 + z^2}} \quad \text{i.e. } u = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$$

Then,

$$\begin{aligned} \vec{F} = -K \nabla u &= \left\langle \frac{KCx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{KCy}{(x^2 + y^2 + z^2)^{3/2}}, \frac{KCz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle \\ &= \frac{cK}{(x^2 + y^2 + z^2)^{3/2}} \cdot \langle x, y, z \rangle \end{aligned}$$

The unit normal vector of a sphere (see lecture notes) is $\hat{n} = \frac{1}{a} \langle x, y, z \rangle$

$$\begin{aligned} \text{Then, } \vec{F} \cdot \hat{n} &= \frac{cK}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle \cdot \frac{1}{a} \langle x, y, z \rangle = \frac{cK}{a} \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{cK}{a} \cdot \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{cK}{a} \cdot \frac{1}{a} = \frac{cK}{a^2} \quad \left(\text{Since on } S \right. \\ &\quad \left. x^2 + y^2 + z^2 = a^2 \right) \end{aligned}$$

$$\text{Then, } \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \frac{cK}{a^2} \, dS = \frac{cK}{a^2} \underbrace{\iint_S 1 \cdot dS}_{\text{surface area}} = \frac{cK}{a^2} \cdot 4\pi a^2 = 4\pi cK$$

6)

$$\text{a) } \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & z \end{vmatrix} = (x-y)\hat{i} - y\hat{j} + \hat{k}$$

Here S is the portion of the plane $3x+2y+z=1$ over

$$D = \left\{ (x,y) \mid 0 \leq x \leq \frac{1}{3}, 0 \leq y \leq \frac{1}{2}(1-3x) \right\}$$

• Orient S upward so that C has positive orientation.

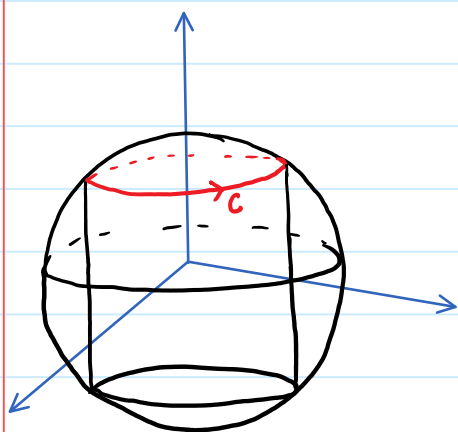
Then by Stokes Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \quad \text{and since } 3x+2y+z=1 \text{ can be rewritten as } z=g(x,y)=1-3x-2y,$$

we have formula for flux of vector field over surface S w/ outward unit normal (see Ques 4)

$$\begin{aligned} &= \iint_D \left[-(x-y) \cdot (-3) - (-y)(-2) + 1 \right] dA = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{2}(1-3x)} (1+3x-5y) \, dy \, dx \\ &= \int_0^{\frac{1}{3}} \left([1+3x]y - \frac{5y^2}{2} \right) \Big|_{y=0}^{\frac{1-3x}{2}} dx = \int_0^{\frac{1}{3}} \left[\frac{1}{2}(1-3x)(1+3x) - \frac{5}{2} \cdot \frac{1}{4}(1-3x)^2 \right] dx \\ &= \int_0^{\frac{1}{3}} \left(-\frac{81}{8}x^2 + \frac{15}{4}x - \frac{1}{8} \right) dx = \left[-\frac{27}{8}x^3 + \frac{15}{4}x^2 - \frac{1}{8}x \right]_0^{\frac{1}{3}} = \frac{1}{24} \end{aligned}$$

$$\text{b) } \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} \quad \text{by Stokes' Theorem.}$$



To find the boundary curve C , solve
 $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 = 4$
 $\Rightarrow z^2 = 12 \Rightarrow z = \sqrt{12}$ (since $z \geq 0$)

• So C is given by $x^2 + y^2 = 4$ and $z = \sqrt{12}$.

Note that we are free to choose an oriented surface S as long as it has C as its boundary, so pick S to be $x^2 + y^2 \leq 4$, $z = \sqrt{12}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y^3 & 1 & z \end{vmatrix} = -3x^2 y^2 \hat{k}$$

Note that surface S is given by $\vec{r}(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{12})$
 with parameter domain $D = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \hat{k} (r \cos^2 \theta + r \sin^2 \theta) = r \hat{k}$$

and,

$$|\vec{r}_r \times \vec{r}_\theta| = r \quad \text{and therefore} \quad \hat{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \hat{k}$$

• Note that for us to traverse C counterclockwise when viewed from above we need to choose $\hat{n} = \langle 0, 0, 1 \rangle$.

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(r, \theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) dr d\theta = \int_0^{2\pi} \int_0^2 -3r^4 \cos^2 \theta \sin^2 \theta \hat{k} \cdot r \hat{k} dr d\theta$$

$$\begin{aligned}
&= -3 \int_0^{2\pi} \int_0^2 r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \\
&= -3 \int_0^{2\pi} (\cos \theta \sin \theta)^2 \left[\frac{r^6}{6} \right]_0^2 d\theta = -32 \int_0^{2\pi} \frac{\sin^2 2\theta}{4} d\theta \quad (\sin 2\theta = 2 \sin \theta \cos \theta) \\
&= -8 \int_0^{2\pi} \sin^2 2\theta \, d\theta = -8 \int_0^{2\pi} \frac{1 - \cos 4\theta}{2} d\theta = -8 \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8} \right]_0^{2\pi} = -8\pi.
\end{aligned}$$

7) Let S be the surface in the plane $x+y+z=1$ w/ upward orientation enclosed by C .
Then, $S : z = g(x,y) = 1-x-y$ and the upward normal vector \hat{n} given by

$$\hat{n} = \frac{\left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle}{\sqrt{1 + g_x^2 + g_y^2}} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$$

and orient C in the counterclockwise direction as viewed from above.

Note that $\int_C z \, dx - 2x \, dy + 3y \, dz = \int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle z, -2x, 3y \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix} = 3\hat{i} + \hat{j} - 2\hat{k}$$

By Stokes Thm,

$$\begin{aligned}
\int_C z \, dx - 2x \, dy + 3y \, dz &= \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = \iint_S \langle 3, 1, -2 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle dS \\
&= \frac{2}{\sqrt{3}} \iint_S dS = \frac{2}{\sqrt{3}} A(S)
\end{aligned}$$

So the value of line integral is $\frac{2}{\sqrt{3}}$. surface area of region enclosed by C , regardless of shape or it's location.

8)

$$a) E = \left\{ (x, y, z) \mid 0 \leq x \leq a, 0 \leq y \leq b\left(1 - \frac{x}{a}\right), 0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \right\}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(zx) = 1 + x$$

Then by divergence theorem,

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iiint_E \nabla \cdot \vec{F} \, dV = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (x+1) \, dz \, dy \, dx = \int_0^a \int_0^{b(1-\frac{x}{a})} (x+1) \left[c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \right] dy \, dx \\ &= c \int_0^a (x+1) \cdot \left[\left(1 - \frac{x}{a}\right)y - \frac{1}{2b} y^2 \right]_{y=0}^{y=b(1-\frac{x}{a})} dx = \frac{1}{2} bc \int_0^a (x+1) \left(1 - \frac{x}{a}\right)^2 dx \\ &= \frac{1}{2} bc \left[\frac{1}{4a^2} x^4 + \frac{1}{3a^2} x^3 - \frac{2}{3a} x^2 + \frac{1}{2} x^2 - \frac{1}{a} x^2 + x \right]_0^a \\ &= \frac{1}{2} bc \left[\frac{1}{12} a^2 + \frac{1}{3} a \right] = \frac{abc}{24} (a+4) \end{aligned}$$

$$b) \vec{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2} \langle x, y, z \rangle$$

$$\begin{aligned} \text{Then } \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} \left(x \sqrt{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial y} \left(y \sqrt{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial z} \left(z \sqrt{x^2 + y^2 + z^2} \right) \\ &= x \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2x + (x^2 + y^2 + z^2)^{1/2} + y \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2y + (x^2 + y^2 + z^2)^{1/2} \\ &\quad + z \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2z + (x^2 + y^2 + z^2)^{1/2} \\ &= \frac{4(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}} = 4\sqrt{x^2 + y^2 + z^2} \end{aligned}$$

$$\begin{aligned} \text{Then, } \iint_S \vec{F} \cdot d\vec{S} &= \iiint_E 4\sqrt{x^2 + y^2 + z^2} \, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 4\rho \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 4\rho^3 \, d\rho = 1 \cdot 2\pi \cdot 1 = 2\pi \end{aligned}$$

$$9) \nabla \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\begin{aligned} \iiint_E \nabla \cdot \vec{F} \, dV &= \int_{\pi/2}^{\pi} \int_0^{2\pi} \int_0^2 3\rho^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/2}^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 3\rho^4 \, d\rho \\ &= \left[-\cos\phi \right]_{\pi/2}^{\pi} \cdot 2\pi \cdot \left[\frac{3\rho^5}{5} \right]_0^2 = \frac{192\pi}{5} \end{aligned}$$

10) (Disregard this solution, I worked out a different ques)

$$a) \vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\vec{x}}{|\vec{x}|^3}$$

Then,

$$\begin{aligned} \nabla \cdot \vec{F} &= \nabla \cdot \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle \\ &= \frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\ &= \left[\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-2y^2 + x^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-2z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{5/2}} \right] = 0 \end{aligned}$$

Since \vec{F} is not defined at the origin we cannot apply divergence theorem on a region containing $(0,0,0)$.

Let us denote the ellipsoid in question by S_2 and pick a surface S_1 that bounds the origin.

The simplest surface to work with would be a sphere, so let S_1 be a small sphere w/ radius a and center origin (contained inside S_2). Then similar to the example worked out in class we have,

$$0 = -\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} \Rightarrow \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S}$$

Now we can compute the surface integral over S_1 ,

\hat{n} at \vec{x} is $\frac{\vec{x}}{|\vec{x}|}$.

$$\vec{F} \cdot \hat{n} = \frac{\vec{x}}{|\vec{x}|^3} \cdot \left(\frac{\vec{x}}{|\vec{x}|} \right) = \frac{|\vec{x}|^2}{|\vec{x}|^4} = \frac{1}{|\vec{x}|^2} = \frac{1}{a^2} \quad \left(\text{Since } S_1 \text{ is given by } |\vec{x}| = a \right)$$

Then,

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \hat{n} \, dS = \iint_{S_1} \frac{1}{a^2} \, dS = \frac{1}{a^2} \iint_{S_1} dS = \frac{1}{a^2} \cdot 4\pi(a^2) = 4\pi$$

b) Assume $\vec{F} = \nabla \times \vec{G}$ for some \vec{G} whose domain is the same as \vec{F}
i.e. domain \vec{G} is \mathbb{R}^3 - origin.

$$\text{Then, } 4\pi = \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \nabla \times \vec{G} \cdot d\vec{S}$$

and since origin is not on the ellipsoid, we can use Stokes' Thm to say that

$$4\pi = \iint_{S_2} \nabla \times \vec{G} \cdot d\vec{S} = \int_{\partial S_2} \vec{G} \cdot d\vec{r} \quad \text{but this cannot happen since } S_2 \text{ has no boundary.}$$

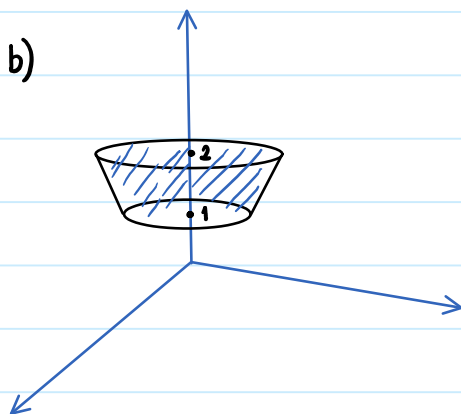
Thus \vec{F} cannot be expressed as the curl of another vector field \vec{G} .

$$11a) \vec{c}(t) = \left(t^2, \frac{t^3}{3} - t \right), -\sqrt{3} \leq t \leq \sqrt{3}$$

Then we know from class notes that using Green's Theorem, that the area of the region D bounded by the curve C is given by

$$\begin{aligned} A(D) &= -\oint y \, dx = -\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{t^3}{3} - t \right) \cdot 2t \, dt \\ &= -\int_{-\sqrt{3}}^{\sqrt{3}} \frac{2t^4}{3} - 2t^2 \, dt = -2 \int_0^{\sqrt{3}} \frac{2t^4}{3} - 2t^2 \, dt \quad \left(\text{Since integrand is even} \right. \\ &\quad \left. \text{and limit of integration is symmetric} \right) \end{aligned}$$

$$= -4 \left[\frac{t^5}{15} - \frac{t^3}{3} \right]_0^{\sqrt{3}} = -4 \left[\frac{9\sqrt{3}}{15} - \frac{3\sqrt{3}}{3} \right] = -4 \left[\frac{3\sqrt{3}}{5} - \frac{5\sqrt{3}}{5} \right] = \frac{8\sqrt{3}}{5} \quad \square$$



Then,

$$E = \{ (\rho, \phi, \theta) \mid 1 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi \}$$

$$\text{Then, } \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} \, dV$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} (x - yz + 1) + \frac{\partial}{\partial y} (y + xz) + \frac{\partial}{\partial z} (z + yx^2) = 3$$

$$\begin{aligned} \text{Then, } \iint_S \vec{F} \cdot d\vec{S} &= \iiint_E 3 \, dV = 3 \int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r \, dr \, d\theta \, dz = 3 \int_1^2 \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{\frac{\pi}{2}} d\theta \, dz = 3 \int_1^2 \int_0^{2\pi} \frac{z^2}{2} d\theta \, dz \\ &= 3 \int_1^2 \pi z^2 \, dz = \pi z^3 \Big|_1^2 = 7\pi \end{aligned}$$

Or alternatively you can use geometry to compute triple integrals i.e.

$$\begin{aligned} \iiint_E 3 \, dV &= 3 \operatorname{vol}(E) = 3 \left[\operatorname{vol} \left(\text{cone with } r=2, h=2 \right) - \operatorname{vol} \left(\text{cone with } r=1, h=1 \right) \right] \\ &= 3 \cdot \left(\frac{\pi (2)^2 \cdot 2}{3} - \frac{\pi (1)^2 \cdot 1}{3} \right) = 7\pi \end{aligned}$$

and $V = \frac{\pi r^2 h}{3}$.